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NUMBER, AND THE FUNDAMENTAL LAWS OF ALGEBRA

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I

HISTORICAL investigation supports the natural inference that we initially arrive at our conception of number through counting. By number we mean the result arrived at by counting. In the first stages, the child associates numbers with objects; and the greatest difficulty is to inculcate the notion of number as an abstract entity. Bernard Shaw once confessed to me that, until he had read Jevons, the economic abstractions of Karl Marx were as meaningless to him as the statement: five apples plus three peaches equals eight pears! But it is clear, nevertheless, that five apples plus three peaches equals something; and indeed eight something. When the child is taught to grasp the abstract number eight as the result of the addition of unlike and disparate objects, the first difficulty has been overcome.

These abstract numbers the mathematician terms positive whole numbers or integers. The contemplation of the world of actual facts, as revealed to us by the senses, supplies us only with this type of number, as the result of counting. There are many other types of number; and it must be understood at the outset that these other numbers and types of numbers are formal inventions of the mathematician. Thus a world of symbols is brought into being through the exercise of the imagination. The question with which we are here concerned is not the philosophical inquiry regarding the reality of such numbers, but the survey of the mode by which this machinery of symbolism is evoked. What is the rationale, the logical basis, for the evocation of such a world of number? How do we justify the introduction of the minus sign, the distinction between cardinal and ordinal number, the formal results of the operations of multiplication, using all different combinations of signs, the use of the symbols $\sqrt{-1}$ and $\sqrt{-a}$, and so on? A mere sketch for survey of some of these basic difficulties and modes for surmounting them through logical reasoning is all that will be attempted here. But the purpose of such a survey is to suggest the great and crying need for an elementary work on number and the fundamental algebraic operations, which may be employed in the upper grades in the high school and during the first year in college.

II

Three fundamental conceptions, of modern origin, should be fully grasped at the outset, to enable the pupil to understand the best contemporary estimates of the nature of number. First there is the notion of *one-to-one correspondence,* uniting objects thrown together in groups or collections. Second, there is the notion of *invariance*—the quality of remaining unchanged through a series of different operations. Third, there is the notion of an underlying generality or universality of law in numerical operation, which may be designated the *permanence of the formal laws.* These three principles, taken in conjunction with our natural conception of enumeration, and supported by a comprehensible extension of the domain of ordinal numbers in conformity with natural conceptions of direction, will enable one to build the substructure upon which the edifice of algebra, and indeed of analysis, properly considered, rests.

First, let us take the notion of one-to-one correspondence. Suppose there is a company of soldiers, each of whom has a gun and a pistol. Here are three groups of objects: soldiers, guns, pistols. Two groups may be brought, as we say, into a one-to-one correspondence if we can combine the elements of the one group with the elements of the other group, mating the diverse elements in pairs. Such groups are called equivalent; and it is obvious that if two groups are equivalent to an unique third group, they are equivalent to one another. Thus each soldier is mated with a gun; also each soldier is mated with a pistol; and accordingly each gun is mated with a pistol, namely the gun and pistol carried by that particular soldier.

Now there is some property which is common to all these groups—which is entirely independent of the character of the elements of which the groups are composed. This property is called its *cardinal number*—that is, the abstract positive integer, as we say, which characterizes the group. This property is arrived at through the comparison and mating in pairs of the elements of two different groups. We make abstraction of that property which is common to the two groups.

Underlying this conception is the second notion mentioned above, that of invariance. For the natural query is raised: Suppose the elements of the group or groups be re-arranged within the group, will not the *cardinal number* of the group thereby be altered? The answer must be in the negative. For if we re-arrange the objects of the group, we shall merely arrive at an equivalent group, placed in a one-to-one correspondence with the group of elements as originally arranged. Hence we may conclude that the cardinal number of a group is the *invariant property* of the group—that is to say, that property of the group which is independent of the character of the objects and of their arrangement within the group. Cardinal number is the property of invariance under permutation of the elements of a group.

III

As soon as we reach the conception of a natural scale of numbers, the ordinal number enters into play. For the cardinal numbers proceed according to a definite system; since we can start with a group containing but a single element, and consecutively adjoin single elements forming new groups with their associated cardinal numbers. If we arrange these numbers, thus consecutively arrived at, in order upon any sort of continuous scale, we have the ordinal system. The ordinal scale may be thought of in connection with motion in either time or space or both simultaneously; and the natural numbers may be regarded as mere names for states of progress—in time, as counting aloud; in space, as moving continuously along a given path; in uniform motion, traversing equal intervals of space in equal intervals of time.

There are no serious difficulties to be encountered in establishing the commutative and associative laws of addition on the basis of the principles already established and the definition of addition. This latter may be stated as follows: to add n to m is to find the number which occupies the *nth* place after m in the natural scale. The laws thus established easily are:

The commutative law: $a + b = b + a$
The associative law: $a + (b + c) = (a + b) + c$

In like manner, the three laws for multiplication of positive integers, commutative, associative, and distributive, may readily be established. These are as follows:

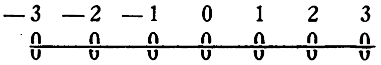
The commutative law: $ab = ba$
The associative law: $a\,(bc) = (ab)\,c$
The distributive law: $a\,(b + c) = ab + ac$

All derive at once from the principles already established, and the definition of multiplication. This latter may be stated as follows: to multiply m by n is to find the sum of n numbers each of which is m.

The introduction of the idea of the negative results in the enlargement of our natural scale. We define subtraction as follows: to subtract n from m is to find the number which occupies the *nth* place before m; and the remainder is designated $m - n$. Hence we reach, by our definition of addition,

$$(m - n) + n = m.$$

As we desire, for purposes of convenience and utility, to extend our natural scale in either direction, at will, we *arbitrarily create or invent* a set of symbols to denote the successive positions on the scale, extended in the direction opposite to that of the natural scale. These symbols we denote by 0, -1, -2, -3, and so on. Thus



According to our conception of group, the symbol 0 may be understood to fall in the class of cardinal numbers. But the other new numbers, -1, -2, -3, and so on, have no cardinal meaning whatsoever.

It should be pointed out, clearly, to the student that the proposition $5 - 8 = 7 - 10$ is not capable of proof. These indicated "differences" acquire equality as the result of the significance we impart to them according to an agreed-upon convention. The second important fact that should be impressed upon the pupil is that the agreed-upon convention conforms to the underlying laws on principles which we have already shown to obtain in the case of positive integers. This will appear more clearly as we proceed with the discussion. We know that the following equation between actual positive integers is true:

$$8 - 5 = 10 - 7$$

We invest with a definite meaning the symbols $5 - 8$, and $7 - 10$; that is: $5 - 8$ represents the eighth number *before* 5 on the scale; and $7 - 10$ represents the tenth number *before* 7 on the scale. These operations, thus carried out, bring us to identically the same position on the scale, viz: the position denoted by the symbol -3.

The universal principle here invoked has been termed by Hankel the *principle of permanence;* Schubert has termed it the *principle of no exception,* and stated it as follows:

In the construction of arithmetic every combination of two previously defined numbers by a sign for a previously defined operation (plus, minus, times, etc.), shall be invested with meaning, even where the original *definition of the operation used excludes such a combination; and the meaning imparted is to be such that the combination considered shall obey the same formula of definition as a combination having from the outset a signification, so that the old laws of reckoning shall still hold good and may still be applied to it.*

The generalization thus phrased, expressing the underlying permanence of the formal laws, has a formidable sound and appearance. It must be studied carefully through means of illustrations for its full comprehension by beginners. The effort will be made to indicate the grounds, in specific cases, for the formulation of such a law.

Let us consider, for example, the symbol $a + (-b)$. How shall we denote the plus sign followed by the minus sign? An appeal to the formal laws established for positive integers may guide us. Thus we know that $a + b = b + a$.

If this is to hold true for our new numbers, we must have

$$-b + b = b + (-b)$$

Hence

$$b + (-b) = 0$$

But

$$b - b = 0;$$

and therefore we may set

$$b + (-b) = b - b.$$

This suggests at once the more general principle:

$$a + (-b) = a - b.$$

Again, in the case of multiplication, we must, in order to retain the formal relation between addition and subtraction, *formulate the new definition*:

$$\begin{aligned}(-a)\ b &= (-a) + (-a) + \text{....to } b \text{ terms}\\ &= -a - a - \text{....to } b \text{ terms}\\ &= -ab\end{aligned}$$

Lastly, let us consider the problem of

$$(-a).\ (-b).$$

Invoking our principle of no exception, we have, since $0 + 0 + \ldots$ to a terms $= 0$, and we must retain the formal law $ab = ba$, the following laws hold: $0.a = 0$, and $a.0 = 0$. Whence it follows that

$$(-a).\ (-b + b) = -a.0 = 0.$$

But since the distributive law $a\ (b + c) = ab + ac$ must formally hold, we have

$$\begin{aligned}(-a).(-b + b) &= (-a)\ (-b) +\\ (-a)\ b &= (-a).(-b) - ab\end{aligned}$$

according to the preceding paragraph.

Hence it follows, axiomatically, that

$$(-a).(-b) - ab = 0.$$

But by definition $ab - ab = 0$. And consequently we are impelled to the *definition* $(-a).\ (-b) = ab$.

In order to make clear these principles and the role which the "categorical imperative" of formalism plays in such matters, let us consider a few simple numerical operations.

On the basis of the definitions for positive integers, we can proceed as follows:

$$(7 - 4)\ 5 = 7.5 - 4.5 = 35 - 20 = 15.$$

If, however, we have the indicated operation: $(4 - 7)\ 5$, we must *assume* the distributive law of multiplication to hold for the negative number $(4 - 7)$ and proceed formally as before:

$$(4 - 7) 5 = 4.5 - 7.5 = 20 - 35.$$

In the first case, since $7 - 4 = 3$ we have $3.5 = 15$.

In the second case, since $4 - 7$ is *defined to be* -3, and $20 - 35$ is *defined* to be -15, we have $-3.5 = -15$.

Lastly let us consider such an indicated operation as $(4 - 7)\ (2 - 5)$. Now first consider the following problem: $(7 - 4)\ (5 - 2)$. We man write $(7 - 4)\ (3) = 7.3 - 4.3 = 21 - 12$.

Now if we proceed, *formally,* by the same method we would have

$$\begin{aligned}&(4 - 7)\ (2 - 5) = 7)\ (-3)\\ &= (4)\ (-3) - (7)\ (-3)\\ &= -12 - (-21)\end{aligned}$$

Hence we must, at this point, attach a meaning to the symbol $-(-21)$. If we *define* $-(-a) = +a$ then we have

$$-12 - (-21) = -12 + 21 = 9$$

But $(4 - 7) = -3$, and $2 - 5 = -3$ according to our definitions. Hence the conclusion is reached: $(-3).(-3) = +9$.

What is the justification, then, of the formulation of this principle of no exception? Is it merely the mathematician's ingenious but artificial means of advancing plausible hypotheses for reaching desired conclusions? Or is it, rather, an illustration of the persistence of the reign of law in the realm of formal logic? Perhaps the answer may be phrased as follows, for the benefit of the elementary student of numbers: *Those laws are true which, however endlessly applied and invoked, involve and lead to no contradictions.* Under the reign of his majesty, the principle of no exception, number conforms adequately to the results derived from human experience. The experiential test, then, at least for the beginner, may be regarded as the criterion of validity for the principle of no exception.

In the next paper in this series, the appeal to the imagination through the eye will be made. Various images and pictorial devices for elucidating and demonstrating certain of the formal laws will be given in considerable fulness and detail.

(To be Continued)